Three Cards Suffice

Colm Mulcahy

Figure 1. The middle card is the nine of diamonds.

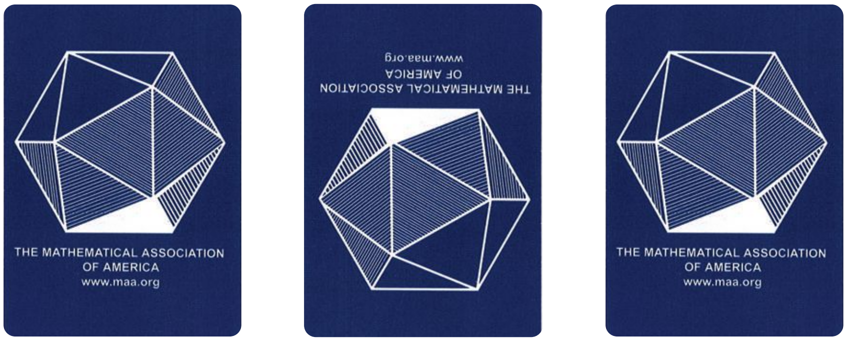

Figure 2. The rightmost card is the king of hearts.

More than 65 years ago, William Fitch Cheney Jr. (1904–1974) conceived one of the greatest mathematical card tricks [2], and since the 1980s it has found new popularity thanks to the efforts of Art Benjamin, Elwyn Berlekamp, Paul Zorn, and others. It goes like this:

A spectator chooses any five cards from a regular 52-card deck. The magician looks at the cards, hides one, and displays the other four face up in a row on the table. Her accomplice enters the room, surveys the cards, and quickly names the hidden card.

There are absolutely no physical or verbal cues; indeed, the trick, originally published under the name "Telephone Stud," was intended to be carried out over the phone, the accomplice only hearing the list of displayed cards from a neutral party.

In recent decades, this trick has been generalized to work with decks of up to 124 cards [1, 6] or a 64-card deck combined with a flip of a coin, in which the accomplice names the fifth card and reveals whether the coin came up heads or tails [1].

In 1999, this trick was generalized in a different direction, as follows [3, 4, 5]:

The magician is given any four cards from a regular 52-card deck. She looks at the cards, hides one, then displays the other three in a row on the table, some face up, some face down. Her accomplice enters the room and names the hidden card.

The trick works for any four cards. The accomplice can even name the hidden card, which we often refer to as the target card, when all three displayed cards are face down.

Clearly there is only one way to go from there, and we hereby offer for your amusement the following variation:

The magician is given any three cards from an MAA 52-card deck. She looks at the faces and lays out the cards in a row, leaving at least one card face down. Her accomplice enters the room and names the rightmost face-down card.

For instance, if the magician lays out the cards as in figure 1, the accomplice would identify the middle card as the nine of diamonds. If the cards were presented as in figure 2, the accomplice would state that the rightmost card is the king of hearts.

How the Trick Works

How is such a trick possible?

Readers who are familiar with the five- and four-card versions should expect (1) to use the pigeonhole principle to help determine which of the selected cards is the target card and (2) to use the displayed cards as a code to identify the target card. Surely there is not enough information with only three cards, one of which is the target card!

Or is there?

The use of a deck such as an MAA deck as shown is critical: The card backs are not rotationally symmetric, so the displayed card back can be oriented in one of two distinguishable ways. The use of one-way decks like this in card magic has a long and respectable history.

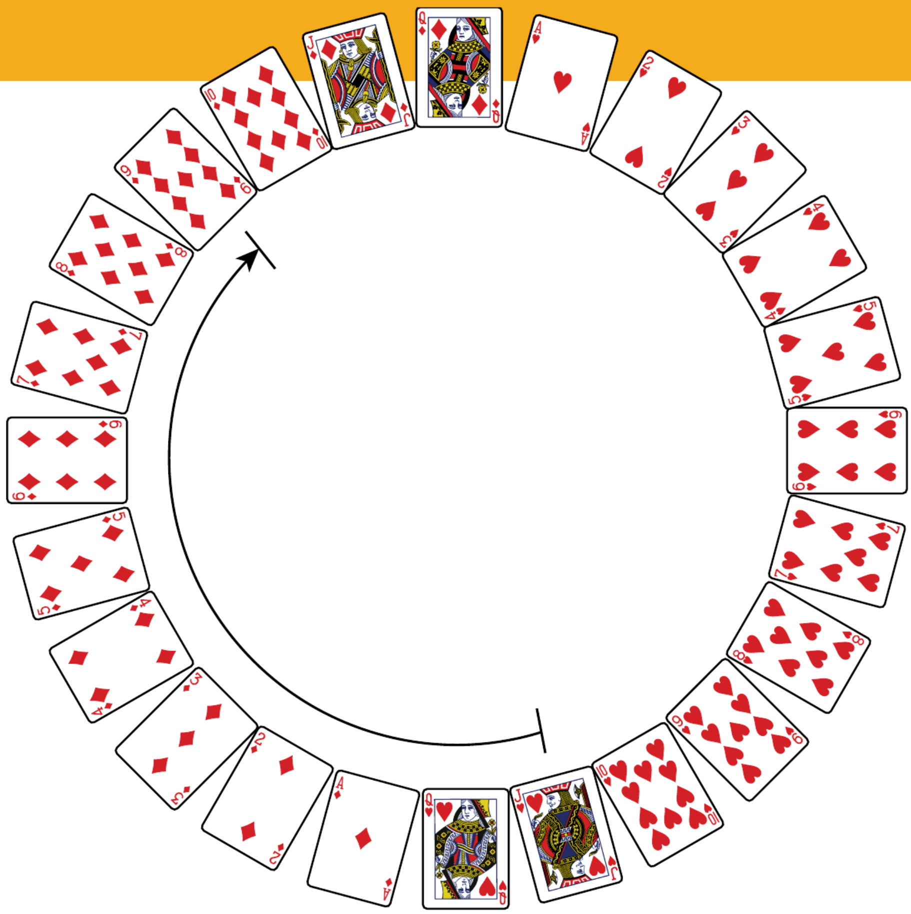

Figure 3. In the red-faced 24-hour clock, the 9D is 10 cards past the JH.

Number to add	Display
1	F – M – M
2	F – M – W
3	F – W – M
4	F – W – W
5	M – F – M
6	M – F – W
7	W – F – M
8	W – F – W
9	M – M – F
10	M – W – F
11	W – M – F
12	W – W – F

Table 1. Convention for an MAA (or one-way) deck.

From this point onward we will express face-up cards by giving the card's value and suit. We will let **M** denote the back of any MAA card that is oriented right side up, and **W** denote the back of any card oriented upside down. So we would express the row of cards in figure 1 as **M** – **W** – **JH**.

The convention is that all three cards are displayed face down if, and only if, one card is a king. The magician chooses one such king as the target card and places it last, face down. It doesn't matter if its MAA logo is right side up or upside down. The orientations of the first two cards convey the target king's suit: **MM** for clubs, **MW** for hearts, **WM** for spades, and **WW** for diamonds (we're suggesting **C****H**a**S**e**D** order for the suits, but alphabetical **CDHS** works also).

For instance, suppose the three cards are the **KH**, **4S**, and **9D**. Then the magician would lay out all three cards face down as **M** – **W** – **M** (as in figure 2) or **M** – **W** – **W**. Either way, her accomplice would correctly identify the final card as the **KH**.

Now, suppose none of the cards is a king. Then it is as if the magician is working with a 48-card deck. The pigeonhole principle guarantees that two of the three chosen cards must have the same color. Without loss of generality, assume there are two red cards. If all three cards are red, the magician just focuses on two of them for what follows.

Imagine the red cards in a king-less deck arranged in a giant 24-hour clock as in figure 3. The two selected red cards must be within 12 hours of each other. Counting clockwise, the magician considers the second card to be the target card. (If the cards have the same value, either one may be taken to be the target card.)

For instance, suppose the three cards are the **JH**, **4S**, and **9D**. The **9D** is 10 cards past the **JH** (bearing in mind that $10 = 12 - (11 - 9)$). So the **9D** is the target card. It and the **4S** will be displayed face down, and the **JH** will be displayed face up. Now the magician must position the three cards in such a way as to convey to her accomplice to "add 10"—as in, "go 10 hours past the **JH** in the 24-hour clock."

In general, the magician places one card face up and must communicate the number between 1 and 12 to add to this card to point to the identity of the rightmost face-down card. The face-up card (denoted by **F**) can be in any of three positions, and the other two can be oriented **M** or **W**. That yields $3 \times 2 \times 2 = 12$ possibilities, as desired. The convention displayed in table 1 is easy to master.

So, starting with the **JH**, **4S**, and **9D**, the magician displays **M** – **W** – **JH**, as shown in figure 1. The presence of a face-up card tells the accomplice that the

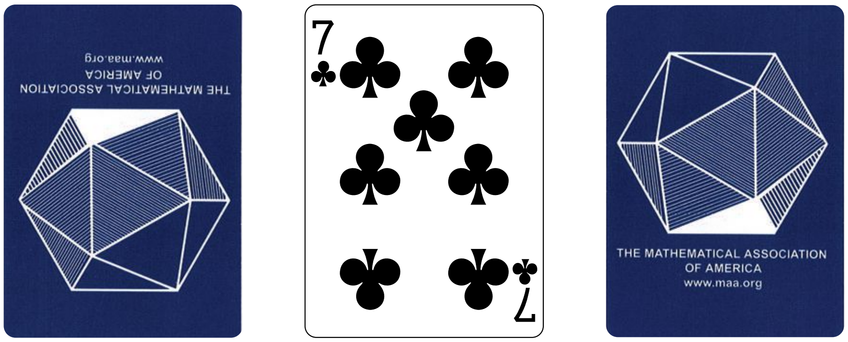

Figure 4. What is the target card?

target card is not a king, and he uses the code in the table to determine that he must go 10 cards past the **JH** in the 24-hour clock. He correctly concludes that the middle card is the **9D**.

To test that you understand the trick, determine the location and identity of the target card in figure 4 (the answer is given at the end of the article).

Two-Way Deck

Surprisingly, we can push this a little further. It can also be performed using an ordinary two-way deck. The trick goes like this:

The magician is given any three cards from a regular 52-card deck. Her accomplice watches her lay out the cards in a row, leaving at least one card face down. The magician points to one of the face-down cards, and her accomplice correctly identifies it.

This setup allows for a little more information to be conveyed to the accomplice. He can see whether the magician lays the cards down from left to right or from right to left. (This flexibility is not without precedent; see the family of Erdős-inspired tricks at [5].) Also, unlike earlier, the magician points to the target card; it need not be the rightmost face-down card.

Number to add	Deal direction	Display
1	L→R	B – F – T
2	L→R	B – T – F
3	L→R	F – B – T
4	L→R	F – T – B
5	L→R	T – B – F
6	L→R	T – F – B
7	R→L	B – F – T
8	R→L	B – T – F
9	R→L	F – B – T
10	R→L	F – T – B
11	R→L	T – B – F
12	R→L	T – F – B

Table 2. Convention for an ordinary (or two-way) deck.

Let's again start by dispensing with the case in which at least one of the three cards is a king, once more agreeing that all three cards are displayed face down if, and only if, one card is a king. The magician chooses one such king as the target card, and then (1) deals left to right if, and only if, it's black, and (2) deals it first if it's a "roundie" (club or heart), and deals it last if it's a "sharpie" (spade or diamond).

The magician points to the target card and requests its identity. The accomplice knows which king it is, having paid attention to which way the cards were dealt and having noted whether the target card was dealt first or last.

Suppose none of the cards is a king, and, without loss of generality, two are red. They are within 12 of each other on the red-faced 24-hour clock. Counting clockwise, the magician considers the second card to be the target card, which we denote **T**. She will present the first card face up, and we denote it **F**. The remaining card will be placed face down (we denote it **B**, as it is the back of the card). Now, she must communicate to her accomplice the number between 1 and 12 to add to the value of **F** to give the identity of the target card.

The cards can be dealt two ways: left to right (L→R) or right to left (R→L). There are six ways to order the cards; recall that the magician will point to the target card, so the accomplice can differentiate between configurations such as **F** – **T** – **B** and **F** – **B** – **T**. We obtain 12 possibilities overall, as desired, suggesting the convention in table 2. Note: To make the table easier to remember, the first six and last six cards in the third column are listed in alphabetical order.

Suppose, as before, the magician is given the **JH**, **4S**, and **9D**. The target card is the **9D**. To communicate to her accomplice that it is 10 cards past the face-up **JH**, she deals **JH** – **T** – **B** right to left. The magician points to **T** and asks the accomplice to name it.

As a final challenge, suppose the magician deals the cards in figure 5 left to right. She points to the rightmost card. What card is it? The answer is given on the next page. ■

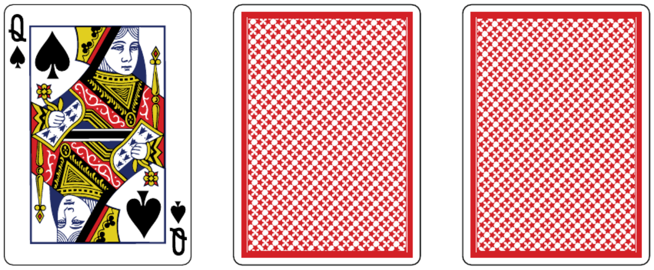

Figure 5. What is the target card?

Further Reading

[1] Michael Kleber, "The best card trick," *Mathematical Intelligencer* 24, no. 1 (Winter 2002): 9–11.

[2] W. Wallace Lee, *Math Miracles*, 1976 ed., Calgary: Micky Hades International, 1951.

[3] Colm Mulcahy, "Fitch Cheney's five card trick," *Math Horizons* 10 (February 2003): 11–13.

[4] Colm Mulcahy, "Fitch four glory," *Card Colm,* February 2005, *www.maa.org/community/maa-columns/past-columns-card-colm/fitch-four-glory.*

[5] Colm Mulcahy, "Mathematical card tricks," *What's New in Mathematics,* October 2000, *www.ams.org/featurecolumn/archive/mulcahy1.html.*

[6] Shai Simonson and Tara Holm, "Using a card trick to teach discrete mathematics," *PRIMUS* XIII, no. 3 (September 2003): 248–269.

Colm Mulcahy is professor of mathematics at Spelman College, Atlanta, Georgia, and author of Mathematical Card Magic: Fifty-Two New Effects *(A K Peters/CRC, 2013) (see review, page 28). He just completed 10 years of writing the bimonthly* Card Colm *column on mathematical card tricks,* cardcolm-maa.blogspot.com.

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Solutions

1. The last card is the target; it is the 2S.

2. The card is the 3C.

Dürer from page 11.

distortion as a fact particular to this picture, making the picture itself wrong, no matter what.

On the other hand, a deeper understanding of geometry can help us to put ourselves, as viewers, "in the right." That is, we can correct the mistakes that other observers like Ivins and Maynard have made; we can see the effect that the master geometer Albrecht Dürer intended. If you view *St. Jerome in His Study* as indicated in figure 6, you'll see that the engraving takes on an amazing realism and depth. The gourd in the picture seems to hover over your head; you feel you could stick your hand in the space under the table; the bench off to the left invites you to come sit down and fluff up the pillows.

And to see this masterpiece come alive, to move into a space that was created centuries ago—to our mind, that's the perfect way to pay homage to a great mathematician and artist on this 500-year anniversary! ■

References

[1] A. Dürer, *Unterweysung der Messung mit dem Zirckel und Richtscheyt (Instruction in Measurement with Compass and Ruler)* (Nuremberg, Germany: Koberger, 1525).

[2] ———, *Die Proportionslehre (The Theory of Proportion)* (Nuremberg, 1528).

[3] W. Ivins, *On the Rationalization of Sight: de artificiali Perspectiva* (New York: Da Capo Press, 1973).

[4] M. Frantz and A. Crannell, *Viewpoints: Mathematical Perspective and Fractal Geometry in Art* (Princeton: Princeton University Press, 2011).

[5] M. Kemp, *The Science of Art* (New Haven: Yale University Press, 1990).

[6] P. Maynard, *Drawing Distinctions* (New York: Cornell University Press, 2005).

Annalisa Crannell is a professor of mathematics at Franklin & Marshall College who is active in developing materials on mathematics and art. She especially enjoys talking to nonmathematicians who haven't (yet) learned where the most beautiful aspects of the subject lie.

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